

A Simple Quantifier-free Formula of Positive Semidefinite Cyclic Ternary Quartic Forms

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Abstract

Quantifier elimination of positive semidefinite cyclic ternary quartic forms is studied in this paper. We solve the problem by the theory of *complete discrimination systems*, function `RealTriangularize` in Maple15 and the so-called *Criteria on Equality of Symmetric Inequalities method*. The equivalent simple quantifier-free formula is proposed and is difficult to obtain automatically by previous methods or quantifier elimination tools.

Key words: Positive semidefinite, quantifier-free formula, ternary quartic.

1. Introduction

The elementary theory of real closed fields (RCF) is expressed in a formal language with atomic formulas of the forms $A = B$ and $A > B$, where A and B are multivariate polynomials with integer coefficients. The problem of quantifier elimination (QE) for RCF can be expressed as: for a given formula of RCF, find an equivalent formula containing the same free (unquantified) variables and no quantifiers.

QE problem is what many researchers have contributed to, including A. Tarski, who gave a first quantifier elimination method for real closed fields in the 1930s, although its publishing delayed for nearly 20 years (Ta48), and G. E. Collins, who introduced a so-called cylindrical algebraic decomposition (CAD) algorithm for QE problem in the 1970s (Co75), which has turned into one of the main tools for QE problems, along with its improved variations. Over the years, new algorithm and important improvements on CAD have appeared, including, for instance, (ACM84b; ACM88; Mc88; Hong90; CH91; Hong92) and (Co98; Mc98; Wei98; Br01a; Br01b; BM05; MB09; Br12). Most of the works, including Tarskis algorithm, were collected in a book (CJ98).

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Many researchers have studied a special quantifier elimination problem (see, for example, (AM88; La88; CH91; Wei94)),

$$(\forall x \in \mathbb{R})(x^4 + px^2 + qx + r \geq 0),$$

called *quartic problem* in the book just mentioned. There are also many researchers that have studied some special QE problems in other ways. In 1987, Choi etc. obtained the necessary and sufficient condition for the positive semidefiniteness of a symmetric form of degree 3 with n variables (CLR87). In 1996, Yang etc. proposed the theory of *complete discrimination systems* for polynomials to discuss the root classification problem of one variable polynomial with real parameters (YHZ96; Yang99). In 1999, Harris gave a necessary and sufficient condition for the positive semidefiniteness of a symmetric form of degree 4 and 5 with 3 variables (Ha99). In 2003, Timofte considered the necessary and sufficient condition for the positive semidefiniteness for symmetric forms of degree d with n variable in $\mathbb{R}^n (d \leq 5)$ (Ti03; Ti05). Applying Timofte's result and the theory of *complete discrimination systems*, Yao etc. obtained a quantifier elimination of the positive semidefiniteness for symmetric forms of degree d with n variable in $\mathbb{R}^n (d \leq 5)$ (YF08). However, the above results are for symmetric forms. Therefore, Han discussed the positive semidefiniteness for more general forms with n variables, including symmetric forms and cyclic forms (Han11).

In this paper, we consider a quantifier-free formula of positive semidefinite cyclic ternary quartic forms, namely the quantifier-free formula of

$$(\forall x, y, z \in \mathbb{R})[F(x, y, z) = \sum_{cyc} x^4 + k \sum_{cyc} x^2 y^2 + l \sum_{cyc} x^2 yz + m \sum_{cyc} x^3 y + n \sum_{cyc} xy^3 \geq 0],$$

which is similar to yet also more complex than *quartic problem*. It is difficult to get an answer directly by previous methods or QE tools. Recall Hilbert's 1888 theorem that says, every positive semidefinite ternary quartic (homogeneous polynomial of degree 4 in 3 variables) is a sum of three squares of quadratic forms (Hilbert88). Hilbert's proof is non-constructive in the sense that it gives no information about the production of an equivalent quantifier-free formula. Notice that $F(x, y, z) \geq 0$ for $x, y, z \in \mathbb{R}$ is equivalent to the following inequality

$$(\forall x, y, z \in \mathbb{R})[f(x, y, z) = \sigma_1^4 + B\sigma_1^2\sigma_2 + C\sigma_2^2 + D\sigma_1\sigma_3 + E\sigma_1 \sum_{cyc} x^2 y \geq 0],$$

where $\sigma_1 = x + y + z$, $\sigma_2 = xy + yz + zx$, $\sigma_3 = xyz$ and B, C, D, E satisfying

$$k = 2B + C + E + 6, l = 2C + D + E + 12 + 5B, m = B + 4, n = B + E + 4.$$

Han (Han11) obtained the following necessary and sufficient condition of $f(x, y, z) \geq 0$,

$$(\forall m \in \mathbb{R})[f(m, 1, km + 1 - k) \geq 0],$$

where k is a real root of the equation

$$Ek^3 - Dk^2 - 3Ek + Dk + E = 0.$$

However, it is still difficult to get a quantifier-free formula by previous methods or QE tools.

Han developed several other methods to solve cyclic and symmetric inequalities including the so-called *Criteria on Equality of Symmetric Inequalities method* (Han11). These

methods can solve a class of QE problems. This paper is firmly rooted in Han's book (Han11), especially the technique dealing with the cyclic and symmetric inequalities. In order to be self contained, we will prove some results in this paper. In order to obtain a simple quantifier-free formula, function `RealTriangularize`(CDMMXX10) of `RegularChains` package in `Maple15` is used to prove inequalities and illustrate semi-algebraic systems without real solution. We also need the theory of *complete discrimination systems* for root classification.

The rest of the paper is organized as follows. Section 2 introduces some basic concepts and results about *complete discrimination systems* for polynomials. Section 3 presents our solution to the positive semidefinite cyclic ternary quartic form.

2. Preliminaries

Given a polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

we write the derivative of $f(x)$ as

$$f'(x) = 0 \cdot x^n + na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1}.$$

Definition 1. (YHZ96; Yang99) (*discriminant matrix*) The Sylvester matrix of $f(x)$ and $f'(x)$

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} \\ a_0 & a_1 & \cdots & a_{n-1} & a_n \\ 0 & na_0 & \cdots & 2a_{n-1} & a_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & a_0 & a_1 & \cdots & a_n \\ & & & 0 & na_0 & \cdots & a_{n-1} \end{vmatrix}$$

is called the *discrimination matrix* of $f(x)$, and denoted by $Discr(f)$.

Definition 2. (YHZ96; Yang99) (*discriminant sequence*) Denoted by D_k the determinant of the submatrix of $Discr(f)$ formed by the first $2k$ rows and the first $2k$ columns. For $k = 1, \dots, n$, we call the n -tuple

$$\{D_1(f), D_2(f), \dots, D_n(f)\}$$

the *discriminant sequence* of polynomial $f(x)$.

Definition 3. (YHZ96; Yang99) (*sign list*). We call list

$$[\text{sign}(D_1(f)), \text{sign}(D_2(f)), \dots, \text{sign}(D_n(f))]$$

the *sign list* of the *discriminant sequence* $\{D_1(f), D_2(f), \dots, D_n(f)\}$

Definition 4. (YHZ96; Yang99) (*revised sign list*). Given a *sign list*

$$[s_1, s_2, \dots, s_n],$$

we construct a new list

$$[\epsilon_1, \epsilon_2, \dots, \epsilon_n]$$

as follows (which is called the *revised sign list*): if $[s_1, s_2, \dots, s_n]$ is a section of the give list, where $s_i \neq 0$, $s_{i+1} = s_{i+2} = \dots = s_{i+j-1} = 0$, $s_{i+j} \neq 0$, then we replace the subsection

$$[s_{i+1}, s_{i+2}, \dots, s_{i+j-1}]$$

by

$$[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \dots],$$

i.e., let

$$\epsilon_{i+r} = (-1)^{\lfloor \frac{r+1}{2} \rfloor} \cdot s_i$$

for $r = 1, 2, \dots, j-1$. Otherwise, let $\epsilon_k = s_k$ i.e., no changes for other terms.

Lemma 5. (YHZ96; Yang99) *Given a polynomial with real coefficients, $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$. If the number of the sign changes of the revised sign list of*

$$\{D_1(f), D_2(f), \dots, D_n(f)\}$$

is v , then the number of the pairs of distinct conjugate imaginary root of $f(x)$ equals v . Furthermore, if the number of non-vanishing members of the revised sign list is l , then the number of the distinct real roots of $f(x)$ equals $l - 2v$.

Theoretically, we can get a quantifier-free formula of the positive semidefinite cyclic ternary quartic form by *complete discrimination systems for polynomials*. Actually, it is impossible because of the complexity.

3. Main result

Lemma 6. (Han11) *Let $x, y, z \in \mathbb{C}$, $x + y + z = 1$ and $xy + yz + zx, xyz \in \mathbb{R}$. The necessary and sufficient condition of $x, y, z \in \mathbb{R}$ is $xyz \in [r_1, r_2]$, where*

$$r_1 = \frac{1}{27}(1 - 3t^2 - 2t^3), r_2 = \frac{1}{27}(1 - 3t^2 + 2t^3)$$

and $t \geq 0$.

Proof. *We consider the polynomial*

$$f(X) = X^3 - (x + y + z)X^2 + (xy + yz + zx)X - xyz,$$

it is obvious that x, y, z are three roots of $f(X) = 0$. By Lemma 5, the equation $f(X) = 0$ has three real roots if and only if

$$D_3(f) \geq 0 \wedge D_2(f) \geq 0,$$

where

$$\begin{aligned} D_2(f) &= (x + y + z)^2 - 3(xy + yz + zx) = 1 - 3(xy + yz + zx), \\ D_3(f) &= (x - y)^2(y - z)^2(z - x)^2 = \frac{1}{27}(4D_2(f)^3 - (3D_2(f) - 1 + 27xyz)^2). \end{aligned}$$

Therefore, using the substitution $t = \sqrt{D_2(f)}$ and $xyz = r$, we have

$$\begin{aligned} x, y, z \in \mathbb{R} &\iff (x-y)^2(y-z)^2(z-x)^2 \geq 0 \wedge (x+y+z)^2 \geq 3xy + 3yz + 3zx, \\ &\iff 4t^6 - (3t^2 - 1 + 27r)^2 \geq 0 \wedge t \geq 0 \\ &\iff \frac{1}{27}(1 - 3t^2 - 2t^3) \leq r \leq \frac{1}{27}(1 - 3t^2 + 2t^3) \wedge t \geq 0. \end{aligned}$$

That completes the proof. \square

Remark 7. Han also get this result by the *Criteria on Equality of Symmetric Inequalities method* (Han11).

We now try to reduce the number of quantifiers of the positive semidefinite cyclic ternary quartic form which is mentioned in the Introduction,

$$(\forall x, y, z \in \mathbb{R})[F(x, y, z) = \sum_{cyc} x^4 + k \sum_{cyc} x^2 y^2 + l \sum_{cyc} x^2 y z + m \sum_{cyc} x^3 y + n \sum_{cyc} x y^3 \geq 0].$$

Lemma 8. (Han11) *The inequality $F(x, y, z) \geq 0$ holds for any $x, y, z \in \mathbb{R}$ if and only if*

$$\begin{aligned} &2 \sum_{cyc} x^4 + 2k \sum_{cyc} x^2 y^2 + 2l \sum_{cyc} x^2 y z + (n+m) \sum_{cyc} x^3 y + (m+n) \sum_{cyc} x y^3 \\ &\geq |(m-n)(x+y+z)(x-y)(y-z)(z-x)| \end{aligned}$$

holds for all $x, y, z \in \mathbb{R}$.

Proof. It is easy to show that for all $x, y, z \in \mathbb{R}$, $F(x, y, z) \geq 0$ is equivalent to: for all $x, y, z \in \mathbb{R}$,

$$\begin{aligned} &2 \sum_{cyc} x^4 + 2k \sum_{cyc} x^2 y^2 + 2l \sum_{cyc} x^2 y z + (n+m) \sum_{cyc} x^3 y + (m+n) \sum_{cyc} x y^3 \\ &\geq (m-n)(x+y+z)(x-y)(y-z)(z-x). \end{aligned}$$

On the other hand, if $F(x, y, z) \geq 0$ holds for any $x, y, z \in \mathbb{R}$, then $F(x, z, y) \geq 0$ also holds for any $x, y, z \in \mathbb{R}$. This inequality is equivalent to

$$\begin{aligned} &2 \sum_{cyc} x^4 + 2k \sum_{cyc} x^2 y^2 + 2l \sum_{cyc} x^2 y z + (n+m) \sum_{cyc} x^3 y + (m+n) \sum_{cyc} x y^3 \\ &\geq (n-m)(x+y+z)(x-y)(y-z)(z-x) \end{aligned}$$

for all $x, y, z \in \mathbb{R}$.

Thus, $F(x, y, z) \geq 0$ for any $x, y, z \in \mathbb{R}$ is equivalent to

$$\begin{aligned} &2 \sum_{cyc} x^4 + 2k \sum_{cyc} x^2 y^2 + 2l \sum_{cyc} x^2 y z + (n+m) \sum_{cyc} x^3 y + (m+n) \sum_{cyc} x y^3 \\ &\geq |m-n|(x+y+z)(x-y)(y-z)(z-x) \end{aligned}$$

for all $x, y, z \in \mathbb{R}$. \square

Theorem 9. *The positive semidefinite cyclic ternary quartic form*

$$\forall x, y, z \in \mathbb{R} \quad F(x, y, z) \geq 0$$

holds if and only if the following inequality holds.

$$(\forall t \in \mathbb{R})[g(t) := 3(2 + k - m - n)t^4 + 3(4 + m + n - l)t^2 + k + 1 + m + n + l - \sqrt{27(m - n)^2 + (4k + m + n - 8 - 2l)^2}t^3 \geq 0].$$

Proof. Substituting $x + y + z, xy + yz + zx, xyz$ with p, q, r , we have

$$|(x-y)(y-z)(z-x)| = \sqrt{(x-y)^2(y-z)^2(z-x)^2} = \sqrt{\frac{4(p^2 - 3q)^3 - (2p^3 - 9pq + 27r)^2}{27}}.$$

We first prove the sufficiency.

If $p = 0$, then the inequality $F(x, y, z) \geq 0$ becomes

$$(2 + k - m - n)q^2 \geq 0.$$

We can deduce $(2 + k - m - n) \geq 0$ from $g(t) \geq 0$ for all $t \geq 0$. (Since $(2 + k - m - n)$ is the leading coefficient of $g(t)$.)

If $p \neq 0$, since the inequality is homogenous, we can assume that $p = 1$. Notice that

$$(x + y + z)^2 \geq 3(xy + yz + zx),$$

thus we have $q \leq \frac{1}{3}$. Using the substitution $t = \sqrt{1 - 3q}$, the inequality $F(x, y, z) \geq 0$ is equivalent to

$$2(2 + k - m - n)t^4 + (16 - 4k + m + n)t^2 - 2 + 2k + m + n + 9(8 - 4k + 2l - m - n)r \geq \sqrt{3}|m - n|\sqrt{4t^6 - (3t^2 - 1 + 27r)^2}.$$

After we compare the above inequality with the desired one, it is sufficient to prove that

$$\begin{aligned} & \sqrt{3}|m - n|\sqrt{4t^6 - (3t^2 - 1 + 27r)^2} \\ & \leq \frac{2\sqrt{27(m - n)^2 + (8 - 4k + 2l - m - n)^2}t^3}{3} + \frac{(8 - 4k + 2l - m - n)(3t^2 - 1 + 27r)}{3}. \end{aligned}$$

After we square both sides and collect terms, the above inequality is equivalent to

$$H^2(r) \geq 0,$$

where

$$H(r) = \frac{2(8 - 4k + 2l - m - n)}{3}t^3 + (3t^2 - 1 + 27r)\sqrt{3(m - n)^2 + \frac{(8 - 4k + 2l - m - n)^2}{9}}.$$

It is obviously true.

So the sufficiency is proved. To prove the necessity, it suffices to show that there exist $x, y, z \in \mathbb{R}$ such that $H(r) = 0$. Notice that

$$\begin{aligned} H(r_1)H(r_2) &= \left(\frac{2(8 - 4k + 2l - m - n)}{3}t^3 - 2t^3\sqrt{3(m - n)^2 + \frac{(8 - 4k + 2l - m - n)^2}{9}}\right) \\ & \quad \left(\frac{2(8 - 4k + 2l - m - n)}{3}t^3 + 2t^3\sqrt{3(m - n)^2 + \frac{(8 - 4k + 2l - m - n)^2}{9}}\right) \\ &= -12t^6(m - n)^2 \leq 0, \end{aligned}$$

where

$$r_1 = \frac{1}{27}(1 - 3t^2 - 2t^3), r_2 = \frac{1}{27}(1 - 3t^2 + 2t^3).$$

Therefore, there exists $r_0 \in [r_1, r_2]$, such that $H(r_0) = 0$. By Lemma 6, such $x, y, z \in \mathbb{R}$ exist and we prove the necessity.

From the above discussion, the theorem is proved. \square

We will apply function `RealTriangularize` of `RegularChains` package in `Maple15` to prove the following lemma.

Lemma 10. *Let $a_0 > 0$, $a_4 > 0$, $a_1 \neq 0$, $a_1, a_2 \in \mathbb{R}$, we consider the following polynomial*

$$f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_4.$$

The discriminant sequence of $f(x)$ is

$$D_f = [D_1(f), D_2(f), D_3(f), D_4(f)],$$

where

$$\begin{aligned} D_1(f) &= a_0^2, \\ D_2(f) &= -8a_0^3a_2 + 3a_1^2a_0^2, \\ D_3(f) &= -4a_0^3a_2^3 + 16a_0^4a_2a_4 + a_0^2a_1^2a_2^2 - 6a_0^3a_1^2a_4, \\ D_4(f) &= -27a_0^2a_1^4a_4^2 + 16a_0^3a_2^4a_4 - 128a_0^4a_2^2a_4^2 - \\ &\quad 4a_0^2a_1^2a_2^3a_4 + 144a_0^3a_2a_1^2a_4^2 + 256a_0^5a_4^3. \end{aligned}$$

For all $x \in \mathbb{R}$, $f(x) \geq 0$ holds if and only if one of the following cases holds,

$$\begin{aligned} (1) & D_4(f) > 0 \wedge (D_2(f) < 0 \vee D_3(f) < 0), \\ (2) & D_4(f) = 0, D_3(f) < 0. \end{aligned}$$

Proof. \implies : If $f(x) \geq 0$ holds for all $x \in \mathbb{R}$, then the number of distinct real roots of $f(x)$ is less than 2. If it equals 2, then the roots of $f(x)$ are all real. If it equals 0, then $f(x)$ has no real root.

If $D_4(f) < 0$ and $D_2(f) > 0$, then the number of non-vanishing members of revised sign list equals $l = 4$. Since $D_4(f)D_2(f) < 0$, then the number of the sign changes of revised sign list equals $v = 1$, thus $l - 2v = 2$. By Lemma 5, the number of distinct real roots of $f(x)$ equals two and the number of the pair of distinct conjugate imaginary root of $f(x)$ $v = 1$, which is impossible. Using function `RealTriangularize`, we can prove that the semi-algebraic system $a_4 > 0, D_4(f) < 0, D_2(f) \leq 0$ has no real solution. Therefore, $D_4(f) \geq 0$. Since $D_1(f) \geq 0$, the number of the sign changes of revised sign list $v \leq 2$.

If $D_4(f) > 0$, thus $l = 4$. Notice that the number of real root of $f(x)$, namely $l - 2v \leq 2$, so $v \geq 1$, from which, we get

$$D_2(f) \leq 0 \vee D_3(f) \leq 0.$$

Using function `RealTriangularize`, we can prove that both the semi-algebraic system $a_4 > 0, a_0 > 0, D_4(f) > 0, D_2(f) \geq 0, D_3(f) = 0, a_1 \neq 0$ and the semi-algebraic system $a_4 > 0, a_0 > 0, D_4(f) > 0, D_3(f) \geq 0, D_2(f) = 0, a_1 \neq 0$ have no real solution. Hence, if $D_4(f) > 0$ and $D_2(f) = 0$, then $D_3(f) < 0$; if $D_4(f) > 0$ and $D_3(f) = 0$, then $D_2(f) < 0$. Thus, when $D_4(f) > 0$, either $D_2(f) < 0$ or $D_3(f) < 0$ holds.

If $D_4(f) = 0$ and $D_3(f) > 0$, then $l = 3$. The number of sign changes of revised sign list v equals either 2 or 0. From $0 \leq l - 2v \leq 2$, we have $v = 1$, which leads to contradiction.

That implies if $D_4(f) = 0$, then $D_3(f) \leq 0$. Using function **RealTriangularize**, we can prove that the semi-algebraic system $a_4 > 0, a_0 > 0, D_4(f) = 0, D_3(f) = 0, a_1 \neq 0$ has no real solution. Hence, when $D_4(f) = 0$, we have $D_3(f) < 0$.

\Leftarrow : If $D_4(f) > 0 \wedge (D_2(f) < 0 \vee D_3(f) < 0)$, then the number of sign changes of revised sign list $v = 2$, so the number of distinct real roots of $f(x)$, $l - 2v$, equals 0, which means for any $x \in \mathbb{R}$, $f(x) > 0$.

If $D_4(f) = 0$ and $D_3(f) < 0$, then $l = 3$, the number of the sign changes of revised sign list $v = 2$. Thus, the number of distinct real roots of $f(x)$, $l - 2v$, equals 1, and the number of the pair of distinct conjugate imaginary root of $f(x)$, v , equals 1, so f has a real root with multiplicity two, which means for any $x \in \mathbb{R}$, $f(x) \geq 0$. \square

Now, we can provide a quantifier-free formula of the positive semidefinite cyclic ternary quartic form.

Theorem 11. *Given a cyclic ternary quartic form of real coefficients*

$$F(x, y, z) = \sum_{cyc} x^4 + k \sum_{cyc} x^2 y^2 + l \sum_{cyc} x^2 y z + m \sum_{cyc} x^3 y + n \sum_{cyc} x y^3,$$

then

$$(\forall x, y, z \in \mathbb{R}) \quad [F(x, y, z) \geq 0]$$

is equivalent to

$$\begin{aligned} & \vee (g_4 = 0 \wedge f_2 = 0 \wedge ((g_1 = 0 \wedge m \geq 1 \wedge m \leq 4) \vee (g_1 > 0 \wedge g_2 \geq 0) \vee (g_1 > 0 \wedge g_3 \geq 0))) \\ & \vee (g_4^2 + f_2^2 > 0 \wedge f_1 > 0 \wedge f_3 = 0 \wedge f_4 \geq 0) \\ & \vee (g_4^2 + f_2^2 > 0 \wedge f_1 > 0 \wedge f_3 > 0 \wedge ((f_5 > 0 \wedge (f_6 < 0 \vee f_7 < 0)) \vee (f_5 = 0 \wedge f_7 < 0))) \end{aligned}$$

where

$$\begin{aligned}
f_1 &:= 2 + k - m - n, f_2 := 4k + m + n - 8 - 2l, \\
f_3 &:= 1 + k + m + n + l, f_4 := 3(1 + k) - m^2 - n^2 - mn, \\
f_5 &:= -4k^3m^2 - 4k^3n^2 - 4k^2lm^2 + 4k^2lmn - 4k^2ln^2 \\
&\quad - kl^2m^2 + 4kl^2mn - kl^2n^2 + 8klm^3 + 6klm^2n + 6klmn^2 \\
&\quad + 8kln^3 - 2km^4 + 10km^3n - 3km^2n^2 + 10kmn^3 - 2kn^4 \\
&\quad + l^3mn - 9l^2m^2n - 9l^2mn^2 + lm^4 + 13lm^3n - 3lm^2n^2 \\
&\quad + 13lmn^3 + ln^4 - 7m^5 - 8m^4n - 16m^3n^2 - 16m^2n^3 - 8mn^4 \\
&\quad - 7n^5 + 16k^4 + 16k^3l - 32k^2lm - 32k^2ln + 12k^2m^2 \\
&\quad - 48k^2mn + 12k^2n^2 - 4kl^3 + 4kl^2m + 4kl^2n - 12klm^2 \\
&\quad - 60klmn - 12kln^2 + 40km^3 + 48km^2n + 48kmn^2 + 40kn^3 \\
&\quad - l^4 + 10l^3m + 10l^3n - 21l^2m^2 + 12l^2mn - 21l^2n^2 \\
&\quad + 10lm^3 + 48lm^2n + 48lmn^2 + 10ln^3 - 17m^4 - 14m^3n \\
&\quad - 21m^2n^2 - 14mn^3 - 17n^4 - 16k^3 + 32k^2l - 48k^2m \\
&\quad - 48k^2n + 80kl^2 - 48klm - 48kln + 96km^2 + 48kmn + 96kn^2 \\
&\quad - 24l^3 - 24l^2m - 24l^2n + 24lm^2 - 24lmn + 24ln^2 - 16m^3 \\
&\quad - 48m^2n - 48mn^2 - 16n^3 - 96k^2 - 64kl + 64km + 64kn + 96l^2 \\
&\quad - 32lm - 32ln - 16m^2 - 32mn - 16n^2 + 64k - 128l + 64m + 64n + 128, \\
f_6 &:= 4k^2 + 2kl - 4km - 4kn + l^2 - 7lm - 7ln + 13m^2 - mn + 13n^2 \\
&\quad - 40k + 20l + 8m + 8n - 32, \\
f_7 &:= -768 + 352k^2 - 332l^2 + 180n^2 + 180m^2 + 56k^3 - 8k^4 \\
&\quad + 14l^3 + 132n^3 + 132m^3 + 42n^4 + 42m^4 - 480k - 60lmn - 192n \\
&\quad + 32klmn - 192m + 912l + l^4 - 354kmn + 158kln + 158klm + 26k^2mn \\
&\quad - 11kln^2 + 22k^2lm + 22k^2ln - 45kmn^2 - 90lm^2n - 45km^2n \\
&\quad - 11klm^2 + 23l^2mn - 90lmn^2 + kl^2m + kl^2n + 36mn - 480km + 592kl \\
&\quad - 480kn - 60lm - 60ln + 8k^3m + 8k^3n - 20k^2l + 32k^2n + 32k^2m \\
&\quad - 12k^3l + 234mn^2 + 234m^2n - 192ln^2 - 258kn^2 - 192lm^2 - 258km^2 \\
&\quad + 116l^2m + 116l^2n + 87m^3n + 87mn^3 - 15kn^3 + 90m^2n^2 - 30ln^3 \\
&\quad - 15km^3 - 30lm^3 + 25l^2m^2 + 25l^2n^2 - 14k^2m^2 - 14k^2n^2 \\
&\quad - 146kl^2 - 10l^3m - 10l^3n - 2k^2l^2 + 3kl^3, \\
g_1 &:= k - 2m + 2, g_2 := 4k - m^2 - 8, g_3 := 8 + m - 2k, g_4 = m - n.
\end{aligned}$$

Proof. By Theorem 9, it suffices to find a quantifier-free formula of

$$\begin{aligned}
(\forall t \in \mathbb{R})[g(t) &:= 3(2 + k - m - n)t^4 + 3(4 + m + n - l)t^2 + k + 1 + m + n + l - \\
&\quad \sqrt{27(m - n)^2 + (4k + m + n - 8 - 2l)^2}t^3 \geq 0].
\end{aligned}$$

Case 1 $\sqrt{27(m - n)^2 + (4k + m + n - 8 - 2l)^2} = 0$, that is $m = n$ and $4k + m + n - 8 -$

$2l = 0$. Hence

$$\begin{aligned} g(t) &= 3(2 + k - 2m)t^4 + 3(4 + 2m - l)t^2 + k + 1 + 2m + l \\ &= 3(2 + k - 2m)t^4 + 3(8 + m - 2k)t^2 + 3(k + m - 1). \end{aligned}$$

If $2 + k - 2m = 0$, then

$$\forall t \in \mathbb{R} \quad g(t) \geq 0 \iff 1 \leq m \leq 4.$$

If $2 + k - 2m > 0$, then

$$\forall t \in \mathbb{R} \quad g(t) \geq 0 \iff (g_1 > 0 \wedge g_2 \geq 0) \vee (g_1 > 0 \wedge g_3 \geq 0).$$

Case 2 $\sqrt{27(m - n)^2 + (4k + m + n - 8 - 2l)^2} \neq 0$ and $1 + k + m + n + l = 0$. In this case, it is easy to show that $2 + k - m - n > 0$. Thus,

$$\begin{aligned} \forall t \in \mathbb{R}, \quad g(t) \geq 0 &\iff \forall t \in \mathbb{R}, \quad 3(2 + k - m - n)t^2 + 3(4 + m + n - l) \\ &\quad - \sqrt{27(m - n)^2 + (4k + m + n - 8 - 2l)^2}t \geq 0 \\ &\iff 27(m - n)^2 + (4k + m + n - 8 - 2l)^2 \leq 36(2 + k - m - n)(4 + m + n - l) \\ &\iff 3(1 + k) \geq m^2 + n^2 + mn. \end{aligned}$$

Case 3 $\sqrt{27(m - n)^2 + (4k + m + n - 8 - 2l)^2} \neq 0$ and $1 + k + m + n + l \neq 0$. In this case, by Lemma 10, we know that for all $x \in \mathbb{R}$, $g \geq 0$ holds if and only if

$$f_1 > 0 \wedge f_3 > 0 \wedge ((f_5 > 0 \wedge (f_6 \leq 0 \vee f_7 \leq 0)) \vee (f_5 = 0 \wedge f_7 < 0)).$$

To summarize, the theorem is proved. \square

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